

Stochastic Modeling of Correlation Radiometer Signals

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Abstract

Many new Earth remote-sensing instruments are embracing both the advantages and added complexity that result from interferometric or fully polarimetric operation. To improve our understanding of calibration options for such instruments, a model of the signals that they measure is presented. A stochastic model is used as it recognizes the non-deterministic nature of any real world measurements while also providing a tractable mathematical framework. A stationary, Gaussian-distributed model structure is proposed. Spectral correlation measures are used to provide a statistical description of the model. A method of realizing the model (necessary for applications such as synthetic calibration-signal generation) is given, and computer simulation results are presented. The signals are constructed using the output of a multi-input, multi-output linear filter system, driven with white noise.

1 Introduction

Radio-interferometers and polarimeters are being used more widely in remote sensing for probing the Earth's lands and oceans (e.g., [1, 2].) While these correlation radiometers have a rich history in the space sciences, earth viewing systems are relatively new. There are distinct differences in the operation of space-viewing radio-telescopes versus orbiting earth-viewing instruments. Therefore, different calibration techniques are needed for orbiting earth-imaging sys-

tems vs. their radio-astronomical counterparts. This paper lays the groundwork for such a technique by examining the signals that these radiometers measure.

While polarimetry and interferometry are usually investigated independently, they are based upon the same idea — measuring the interdependence of two signals. For polarimetry these signals are the vertical and horizontal field amplitudes (or some equivalent pair.) Two-beam interferometry involves measuring the coherency of two signals separated in space and/or time. In order to fully understand these correlation instruments, it is desirable to have an accurate model of the types of signal pairs they measure. This model can then be employed in mathematical analysis, computer simulations, and for the generation of synthetic calibration signals.

Electromagnetic waves generally have some degree of randomness — an unpolarized component in polarimetry or an incoherent component in interferometry. For this reason it is logical to employ a stochastic method when modeling the pair of signals.

2 Background

The following narrowband representation of a polarimetric or interferometric signal pair

(centered on some frequency ω_0) is used.

$$\begin{aligned} X(t) &= \operatorname{Re}\{P(t)e^{i\omega_0 t}\} \\ &= \operatorname{Re}\{(A(t) + iB(t))e^{i\omega_0 t}\} \\ Y(t) &= \operatorname{Re}\{Q(t)e^{i\omega_0 t}\} \\ &= \operatorname{Re}\{(C(t) + iD(t))e^{i\omega_0 t}\} \end{aligned} \quad (1)$$

Where $P(t)$ and $Q(t)$ are complex and $A(t)$, $B(t)$, $C(t)$ and $D(t)$ are real. It should be noted that a narrowband representation does not imply a small fractional bandwidth.

It is assumed that the model is stationary, zero-mean and Gaussian distributed. These assumptions are physically justifiable for the continuum microwave emission observed by earth-sensing radiometers. Since the model is stationary and Gaussian, it is described completely by its second order statistics. These second order statistics are in turn defined by the spectral functions, the Fourier transform of the corresponding temporal-correlation function, i.e.

$$S_{FG}(\omega) = \Im\{E[F(t)G^*(t - \tau)]\}, \quad (2)$$

where $\Im\{\cdot\}$ denotes the Fourier transform.

Appropriate interpretation of the spectral functions $S_P(\omega)$, $S_Q(\omega)$ and $S_{PQ}(\omega)$ allows the model to represent different interferometric or polarimetric scenarios. [3], [4] and [5] can be used to interpret interferometric and polarimetric concepts (such as the visibility function, Stokes parameters and mutual coherence) in terms of the model parameters used in this paper. The Results section includes an example of this.

3 Defining the Model

Equation 1 uses four correlated, real random processes — $A(t)$, $B(t)$, $C(t)$ and $D(t)$. The model can be defined by the statistics of these processes, and as mentioned earlier, is completely defined by the second order statistics of the processes. This paper defines these using the spectral functions (as defined in Equation 2.)

Taking every possible pair of $A(t)$, $B(t)$, $C(t)$ and $D(t)$ (including allowing a process

to be paired with itself) results in ten spectral functions. The goal is to take the spectral functions to be modeled ($S_P(\omega)$, $S_Q(\omega)$ and $S_{PQ}(\omega)$) and use them to determine these ten functions and thus the model.

The assumption of stationarity can be used to reduce the number of independent functions that need to be defined. To ensure the model is stationary, the following relationships must be true.

$$\begin{aligned} S_A(\omega) &= S_B(\omega) \\ S_C(\omega) &= S_D(\omega) \\ S_{AC}(\omega) &= S_{BD}(\omega) \\ S_{AD}(\omega) &= -S_{BC}(\omega) \\ S_{AB}(\omega) &= -S_{AB}(-\omega) \\ S_{CD}(\omega) &= -S_{CD}(-\omega) \end{aligned} \quad (3)$$

Equation 3 shows how the ten defining functions can be defined from a set of six. The last two lines specify an odd-symmetry constraint on $S_{AB}(\omega)$ and $S_{CD}(\omega)$.

It can be shown that the following equations ensure a model with the desired correlation functions ($S_P(\omega)$, $S_Q(\omega)$ and $S_{PQ}(\omega)$.)

$$\begin{aligned} S_A(\omega) &= \frac{1}{4}[S_P(\omega) + S_P(-\omega)] \\ S_C(\omega) &= \frac{1}{4}[S_Q(\omega) + S_Q(-\omega)] \\ S_{AC}(\omega) &= \frac{1}{4}[S_{PQ}(\omega) + S_{PQ}^*(-\omega)] \\ S_{AD}(\omega) &= \frac{i}{4}[S_{PQ}(\omega) - S_{PQ}^*(-\omega)] \\ S_{AB}(\omega) &= \frac{i}{4}[S_P(\omega) - S_P(-\omega)] \\ S_{CD}(\omega) &= \frac{i}{4}[S_Q(\omega) - S_Q(-\omega)] \end{aligned} \quad (4)$$

The above equations completely specify the model and also show that there is a unique, well-defined model for each set of $S_P(\omega)$, $S_Q(\omega)$ and $S_{PQ}(\omega)$ functions. In other words, only by satisfying Equation 4 can $S_P(\omega)$, $S_Q(\omega)$ and $S_{PQ}(\omega)$ be modeled while also ensuring the stationarity of $X(t)$ and $Y(t)$. In addition, it can be shown that the odd-symmetry constraints of Equation 3 are always satisfied.

4 Realizing the Model

The previous sections have developed a stochastic model for a pair of signals. This model is defined by the spectral correlation

functions of $A(t)$, $B(t)$, $C(t)$ and $D(t)$. This is a clear mathematical model that has the potential to be useful in theoretical analysis. However, for applications such as computer simulations and synthetic signal generation it is necessary to create realizations of these random processes. It is sufficient to create realizations of the real variables $A(t)$, $B(t)$, $C(t)$ and $D(t)$, as the other variables ($P(t)$, $Q(t)$, $X(t)$ and $Y(t)$) can be created by deterministic functions of these four realizations.

A common technique in one dimension is to use a linear filter to shape white noise into a desired spectral shape. The problem here is more complicated as the function is from one dimension (time) into four dimensions ($A(t)$, $B(t)$, $C(t)$, $D(t)$). A generalized filter structure with four white, Gaussian, unit variance inputs ($I_1(t)$, $I_2(t)$, $I_3(t)$ and $I_4(t)$) and four outputs ($A(t)$, $B(t)$, $C(t)$ and $D(t)$) is given below (this structure is presented in [6].)

$$\begin{aligned} A(t) &= \sum_{i=1}^4 h_{Ai}(t) * I_i(t) \\ B(t) &= \sum_{i=1}^4 h_{Bi}(t) * I_i(t) \\ C(t) &= \sum_{i=1}^4 h_{Ci}(t) * I_i(t) \\ D(t) &= \sum_{i=1}^4 h_{Di}(t) * I_i(t) \end{aligned} \quad (5)$$

The filter responses must be chosen so that the desired spectral functions are realized. These functions are given by the expression below ('F' and 'G' are dummy variables for the possible outputs 'A', 'B', 'C' and 'D').

$$S_{FG}(\omega) = \sum_{i=1}^4 H_{Fi}(\omega) H_{Gi}^*(\omega) \quad (6)$$

This set of equations is non-linear but the problem can be simplified by writing the system equations in matrix form.

$$\bar{\bar{S}} = \bar{\bar{H}} \bar{\bar{H}}^\dagger \quad (7)$$

Equation 7 uses the matrices defined below, and it must be satisfied at every frequency point (ω dependence has been dropped for clarity.)

$$\bar{\bar{H}} = \begin{bmatrix} H_{A1} & H_{A2} & H_{A3} & H_{A4} \\ H_{B1} & H_{B2} & H_{B3} & H_{B4} \\ H_{C1} & H_{C2} & H_{C3} & H_{C4} \\ H_{D1} & H_{D2} & H_{D3} & H_{D4} \end{bmatrix} \quad (8)$$

$$\bar{\bar{S}} = \begin{bmatrix} S_A & S_{AB} & S_{AC} & S_{AD} \\ S_{AB}^* & S_B & S_{BC} & S_{BD} \\ S_{AC}^* & S_{BC}^* & S_C & S_{CD} \\ S_{AD}^* & S_{BD}^* & S_{CD}^* & S_D \end{bmatrix} \quad (9)$$

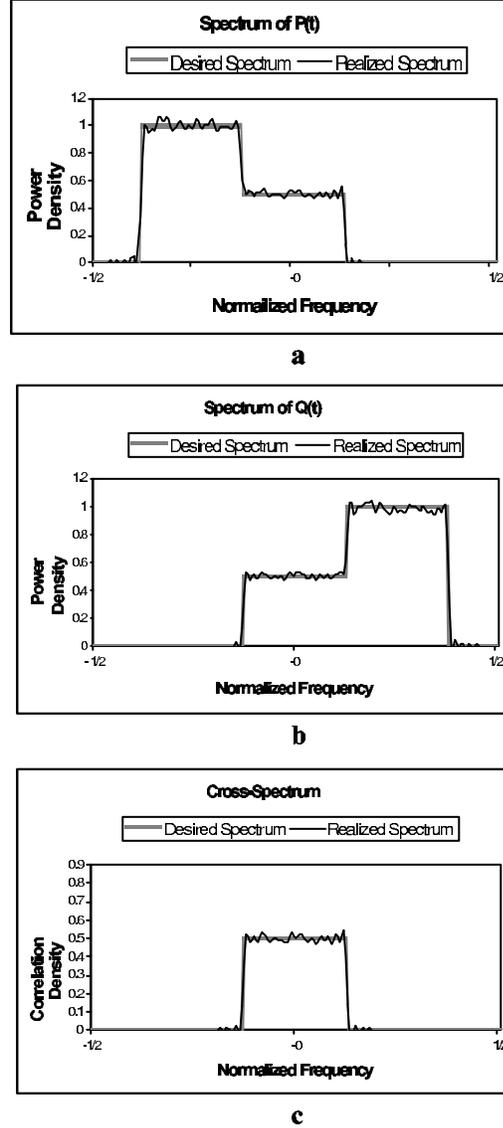


Figure 1: Example of Realized Spectra
a Vertical Field Power Spectrum
b Horizontal Field Power Spectrum
c Cross-Spectrum

For matrices the \dagger operation represents a conjugate transpose. In order to find a suitable set of filters it is sufficient to solve Equation 7 at the frequency points of interest. It is shown in [7] that because $\bar{\bar{S}}$ is positive semi-definite (see [6]), Equation 7 can

always be solved. The method presented in [7] involves diagonalizing the matrix $\overline{\overline{S}}$ and then factorizing it appropriately.

5 Results

A simple example is provided in this section. The wave to be modeled is assumed to have three distinct polarization bands — the lowest band is vertically polarized, the center is linear 45 deg polarized and the highest is horizontally polarized. It is assumed that $P(t)$ represents the vertical channel and $Q(t)$ represents the horizontal channel. $S_P(\omega)$ has a value of 1 in the lowest band, $\frac{1}{2}$ in the center band and 0 in the highest band. $S_Q(\omega)$ is 0 in the lowest band, $\frac{1}{2}$ in the center and 1 in the highest band. There is only interdependence between the two signals in the 45° polarization band, so in the center we have an expected product of $\frac{1}{2}$ for $S_{PQ}(\omega)$.

The realization process was carried out using the MATLAB software package and plots of the resulting spectra can be seen in Figure 1. The figures show the spectra as specified above, and the resulting realization. The estimates of the realized spectra were found using periodogram averaging. 400 spectra were averaged in each case and a rectangular window was used in the time domain to remove the noisy long-lag terms. Each plot has 399 frequency points and the filters were truncated to 199 taps. Figure 1 shows that the resulting spectra agree closely with those specified. The small differences can be accounted for by the necessary truncation of the generation filters and by the fact that a finite number of spectra were used to create the periodogram average.

6 Summary

This paper has presented a model of the signals measured by correlation radiometers (polarimeters and interferometers.) The model is defined by the spectral functions of the two channels. These functions and

the application of the stationary, Gaussian assumption, ensure that the model is well-defined and unique for any set of given spectral functions.

A method of realizing this statistical model was then presented — necessary in applications such as the generation of synthetic calibration signals. The realization method involves generalizing a well-known noise-shaping technique in which a white, Gaussian process is passed through a linear filter in order to color its spectrum to some desired shape. The method was corroborated by presenting results from a MATLAB implementation.

References

- [1] Y. H. Kerr. SMOS: Proposal in answer to the [European Space Agency's] Call for Earth Explorer Opportunity Missions COP-16, November 1998.
- [2] J. R. Piepmeier and A. J. Gasiewski. High-resolution passive microwave polarimetric mapping of ocean surface wind vector fields. *accepted for publication in IEEE Trans. Geosci. Remote Sensing*, July 2000.
- [3] E. Hecht. *Optics*, chapter 9, 11. Addison–Wesley, Reading, Massachusetts, second edition, 1987.
- [4] C. Brosseau. *Fundamentals of Polarized Light: A Statistical Optics Approach*, chapter 3. John Wiley and Sons, New York, 1998.
- [5] S. Pancharatnam. Partial polarization, partial coherence and their spectral descriptions for polychromatic light, part i. In *The Collected Works of S. Pancharatnam*. Oxford University Press, London, 1975.
- [6] G. W. Jenkins and D. G. Watts. *Spectral Analysis and its Applications*, chapter 8, 11. Holden–Day, San Francisco, 1968.
- [7] G. Strang. *Linear Algebra and its Applications*, chapter 6. Academic Press Inc., New York, 1976.